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Complex BPS Domain Walls and Phase Transition in Mass in Supersymmetric QCD

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Abstract

We study the domain walls connecting different chirally asymmetric vacua in supersymmetric QCD. We show that BPS – saturated solutions exist only in the limited range of mass $m \leq m_* \approx 0.8| < \mathrm{Tr}\ \lambda^2 > |^{1/3}$. When $m > m_*$, the domain wall either ceases to be BPS – saturated or disappears altogether. In any case, the properties of the system are qualitatively changed.

1 Introduction

Supersymmetric QCD is the theory involving a gauge vector supermultiplet V and a couple of chiral matter supermultiplets Q^f belonging to the fundamental representation of the gauge group $SU(N)$. The lagrangian of the model reads

$$\mathcal{L} = \frac{1}{2g^2}\mathrm{Tr}\int d^2\theta\ W^2 \ +\ \frac{1}{4}\int d^2\theta d^2\bar{\theta}\ \bar{S}^f e^V S^f - \left(\frac{m}{4}\int d^2\theta\ S^f S_f + \mathrm{H.c.}\right), \quad (1.1)$$

where $S_f = \epsilon_{fg}S^g$ (for further conveniences, we have changed a sign of mass here compared to the standard convention). The dynamics of this model is in many respects similar to the dynamics of the standard (non–supersymmetric) QCD and, on the other hand, supersymmetry allows here to obtain a lot of exact results [1].

Like in the standard QCD, the axial $U_A(1)$ symmetry corresponding to the chiral rotation of the gluino field and present in the tree–level lagrangian (1.1) is broken by anomaly down to Z_{2N}. This discrete chiral symmetry can be further broken spontaneously down to Z_2 so that the chiral condensate $<\mathrm{Tr}\ \lambda^2>$ is formed. There are N different vacua with different phases of the condensate

$$<\mathrm{Tr}\ \lambda^2> = \ \Sigma e^{2\pi i k/N}, \qquad k = 0, \ldots, N-1 \quad (1.2)$$

It was noted recently [2] that on top of N chirally asymmetric vacua (1.2), also a chirally symmetric vacuum with zero value of the condensate exists.

The presence of different degenerate physical vacua in the theory implies the existence of domain walls — static field configurations depending only on one spatial coordinate (z) which interpolate between one of the vacua at $z = -\infty$ and another one at $z = \infty$ and minimizing the energy functional. As was shown in [3], in many cases the energy density of these walls can be found exactly due to the fact that the walls present the BPS–saturated states. The key ingredient here is the central extention of the $N = 1$ superalgebra [3, 4]

$$\{Q^\dagger_{\dot\alpha} Q^\dagger_{\dot\beta}\} = 4\,(\vec{\sigma})_{\dot\alpha\dot\beta}\int d^3x\,\vec{\nabla}\left\{\left[-\frac{m}{2}S^f S_f + \frac{1}{16\pi^2}\mathrm{Tr}\ W^2\right] - \frac{N}{16\pi^2}\mathrm{Tr}\ W^2\right\}_{\theta=0}, \quad (1.3)$$

A domain wall is a configuration where the integral of the full derivative in the RHS of Eq.(1.3) is non–zero so that the standard $N = 1$ SUSY algebra in the wall sector is modified. A BPS–saturated wall is a configuration preserving 1/2 of the original supersymmetry i.e. a configuration annihilated by the action of two certain real linear combinations of the original complex supercharges Q_α [1].

Combining (1.3) with the standard SUSY commutator $\{Q_\alpha, \bar{Q}^{\dot\beta}\} = 2(\sigma_\mu)^{\dot\beta}_\alpha P_\mu$ and bearing in mind that the vacuum expectation value of the expression in the

[1] Such BPS–saturated walls were known earlier in 2–dimensional supersymmetric theories (they are just solitons there) [5] and were considered also in 4–dimensional theories in stringy context [6].

square brackets in Eq.(1.3) is zero due to Konishi anomaly [7] [2], it is not difficult to show that the energy density of a BPS–saturated wall in SQCD satisfies a relation

$$\epsilon \;=\; \frac{N}{8\pi^2}\left|< \mathrm{Tr}\ \lambda^2 >_\infty \;-\; < \mathrm{Tr}\ \lambda^2 >_{-\infty}\right| \tag{1.4}$$

where the subscript $\pm\infty$ marks the values of the gluino condensate at spatial infinities.

The relation (1.4) is valid *assuming* that the wall is BPS–saturated. However, whether such a BPS–saturated domain wall exists or not is a non–trivial dynamic question which can be answered only in a specific study of a particular theory in interest.

In Ref.[4] this question was studied for the $SU(2)$ gauge group in the framework of the effective low energy lagrangian due to Taylor, Veneziano, and Yankielowicz [9]. The situation is particularly simple when the mass parameter m in the lagrangian (1.1) is small compared to $\Lambda_{SQCD} \equiv \Lambda$. In this case, chirally asymmetric vacua are characterized by large values of the matter scalar field χ. The theory involves two different energy scales, and one can integrate out heavy fields and to write the Wilsonean effective lagrangian describing only light degrees of freedom. It is the lagrangian of the Wess–Zumino model with a single chiral superfield X and the superpotential

$$\mathcal{W} = -\frac{2}{3}\frac{\Lambda^5}{X^2} - \frac{m}{2}X^2\,. \tag{1.5}$$

The corresponding potential $U \;=\; |\partial W/\partial\chi|^2$ has two different non–trivial minima at $<\chi^2> \;=\; \pm\chi_*^2 \;=\; \pm\sqrt{4\Lambda^5/3m}$. A domain wall interpolating between these vacua is BPS–saturated. The solution can be found analytically [4]

$$\chi(z) \;=\; \chi_*\frac{1 + ie^{4m(z-z_0)}}{\sqrt{1+e^{8m(z-z_0)}}} \tag{1.6}$$

where z_0 is the position of the wall center.

In this approach, we are not able, however, to detect and study a chirally symmetric vacuum with $<\chi>= 0$ and the corresponding domain wall. Chirally symmetric vacuum appears when taking into account also the degrees of freedom associated with the gluon and gluino fields. The full TVY effective lagrangian is, again, a Wess–Zumino model involving two chiral superfields Φ and X with the superpotential

$$\mathcal{W} = \frac{2}{3}\Phi^3\left[\ln\frac{\Phi^3X^2}{\Lambda^5} \;-\; 1\right] - \frac{m}{2}X^2 \tag{1.7}$$

The corresponding potential for the lowest components ϕ, χ of the superfields Φ, X

$$U(\varphi,\chi) \;=\; \left|\frac{\partial W}{\partial\varphi}\right|^2 + \left|\frac{\partial W}{\partial\chi}\right|^2 \;=\; 4\left|\varphi^2\ln(\varphi^3\chi^2)\right|^2 + \left|\chi\left(m - \frac{4\varphi^3}{3\chi^2}\right)\right|^2 \tag{1.8}$$

[2] See recent [8] for detailed pedagogical explanations.

(in the remainder of this and in the next section we will measure everything in the units of Λ) has three non–trivial minima:

$$\phi \;=\; \chi \;=\; 0 \tag{1.9}$$

$$\begin{aligned}\phi \;&=\; \left(\frac{3m}{4}\right)^{1/6}, \quad \chi \;=\; \left(\frac{4}{3m}\right)^{1/4} \;;\\ \phi \;&=\; e^{-i\pi/3}\left(\frac{3m}{4}\right)^{1/6}, \quad \chi \;=\; i\left(\frac{4}{3m}\right)^{1/4}\end{aligned} \tag{1.10}$$

There are also the minima with inversed sign of ϕ and χ, but they are physically the same as the minima (1.10): the vacuum expectation values of the gauge invariant operators $<\mathrm{Tr}\,\lambda^2> = \;(32\pi^2/3) <\phi^3>$ and $<s^f s_f> = <\chi^2>$ (s^f is the squark field) are the same.

The effective theory (1.5) is obtained when the heavy degree of freedom ϕ is frozen in the Born–Oppenheimer spirit so that $\phi^3\chi^2 = 1$. (In the opposite limit $m \gg 1$, we can freeze instead the heavy matter fields $\chi^2 = 4\phi^3/3m$ and arrive at the Veneziano–Yankielowicz effective lagrangian [10] for pure supersymmetric gluodynamics which involves only the field ϕ.)

Generally, one should study the theory with the potential (1.8). The status of this effective theory is somewhat more uncertain than that of (1.5) — for general value of mass, the TVY effective lagrangian is not Wilsonean; light and heavy degrees of freedom are not nicely separated. But it possesses all the relevant symmetries of the original theory [3] and satisfies the anomalous Ward identities for correlators at zero momenta. We think that the use of the TVY lagrangian is justified as far as the vacuum structure of the theory is concerned.

In Ref.[4] the domain walls interpolating between the chirally symmetric minimum (1.9) and a chirally asymmetric minimum in Eq.(1.10) were studied along these lines. It was shown that these walls are BPS–saturated at any value of mass.

In this paper, we study the complex domain walls interpolating between different minima in (1.10). Rather surprisingly, we have found that the solution of the BPS equation exists only for small enough masses $m \leq m_* = 4.6705\ldots$. At larger values of mass, the BPS equations have no solution. A *phase transition* occurs.

2 Solving BPS equations.

BPS equations for the domain wall in a generalized Wess–Zumino model with two chiral superfields read [5, 3, 4]

$$\partial_z \phi \;=\; \pm \partial \bar{W}/\partial\bar{\phi}, \qquad \partial_z \chi \;=\; \pm\partial\bar{W}/\partial\bar{\chi} \tag{2.1}$$

[3]To see that, one should use the amended potential of Ref.[2] which is "glued" of different sectors related to the different branches of the logarithm; cf. an analogous situation in the Schwinger model [11]. For our present purposes, these complications are irrelevant, however.

In our case, the superpotential is given by the expression (1.7). Let us choose the positive sign in Eq.(2.1) and try to solve it with the boundary conditions

$$
\begin{aligned}
\phi(-\infty) = \left(\frac{3m}{4}\right)^{1/6}, \ \phi(\infty) = e^{-i\pi/3}\left(\frac{3m}{4}\right)^{1/6}, \\
\chi(-\infty) = \left(\frac{4}{3m}\right)^{1/4}, \ \chi(\infty) = i\left(\frac{4}{3m}\right)^{1/4}
\end{aligned}
\tag{2.2}
$$

(the negative sign in (2.1) would describe the wall going in the opposite direction in z). The solution of the equations (2.1) with the boundary conditions (2.2) has the fixed energy which coincides with (1.4) after the proper normalization

$$
\phi^3 \ = \ \frac{3}{32\pi^2}\mathrm{Tr}\ \lambda^2 \tag{2.3}
$$

is chosen.

Technically, it is convenient to introduce the polar variables $\chi = \rho e^{i\alpha}$, $\phi = Re^{i\beta}$. Then the system (2.1) (with the positive sign chosen) can be written in the form

$$
\left\{
\begin{array}{lcl}
\partial_z \rho & = & -m\rho\cos(2\alpha) + \frac{4R^3}{3\rho}\cos(3\beta) \\
\partial_z \alpha & = & m\sin(2\alpha) - \frac{4R^3}{3\rho^2}\sin(3\beta) \\
\partial_z R & = & 2R^2\left[\cos(3\beta)\ln(R^3\rho^2) - \sin(3\beta)(3\beta+2\alpha)\right] \\
\partial_z \beta & = & -2R\left[\sin(3\beta)\ln(R^3\rho^2) + \cos(3\beta)(3\beta+2\alpha)\right]
\end{array}
\right.
\tag{2.4}
$$

The wall solution should be symmetric with respect to its center. Let us seek for the solution centered at $z = 0$ so that

$$
\rho(z) = \rho(-z),\ R(z) = R(-z),\ \alpha(z) = \pi/2 - \alpha(-z),\ \beta(z) = -\pi/3 - \beta(-z) \tag{2.5}
$$

Indeed, one can be easily convinced that the Ansatz (2.5) goes through the equations (2.4).

The system (2.1) has one integral of motion [8]:

$$
\mathrm{Im}\ W(\phi,\chi) \ = \ \mathrm{const} \tag{2.6}
$$

Indeed, we have

$$
\partial_z W = \frac{\partial W}{\partial \phi}\partial_z\phi \ + \ \frac{\partial W}{\partial \chi}\partial_z\chi \ = \ \left|\frac{\partial W}{\partial \phi}\right|^2 \ + \ \left|\frac{\partial W}{\partial \chi}\right|^2 \ = \ \partial_z \bar{W}
$$

It is convenient to solve the equations (2.4) numerically on the half–interval from $z = 0$ to $z = \infty$. The symmetry (2.5) dictates $\alpha(0) = \pi/4$, $\beta(0) = -\pi/6$. The condition (2.6) [in our case Im $W(\phi,\chi) = 0$ due to the boundary conditions (2.2)] implies

$$
\frac{4R^3}{3}\left[\ln(R^3\rho^2) - 1\right] + m\rho^2 \ = \ 0 \tag{2.7}
$$

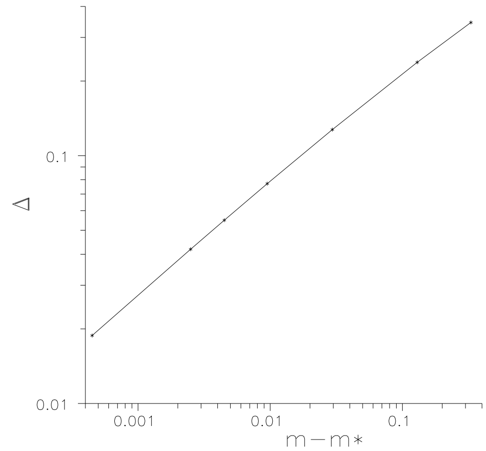

Figure 1: Mismatch parameter Δ as a function of mass

Thus, only one parameter at $z = 0$ [say, $R(0)$] is left free. We should fit it so that the solution would approach the complex minimum in Eq.(1.10) at $z \to \infty$.

It turns out that the solution of this problem exists, but only in the limited range of m. If $m \geq m_* = 4.6705\ldots$, the solution *misses* the minimum no matter what the value of $R(0)$ is chosen. This is illustrated in Fig. 1 where the "mismatch parameter"

$$\Delta \;=\; \min_{R(0)} \min_{z} \sqrt{\left|\chi(z) - i(4/3m)^{1/4}\right|^2 + \left|\phi(z) - e^{-i\pi/3}(3m/4)^{1/6}\right|^2} \tag{2.8}$$

is plotted (in a double logarithmic scale) as a function of mass. The dependence $\Delta(m)$ fits nicely the law

$$\Delta(m) \;=\; 0.56(m - m_*)^{0.44} \tag{2.9}$$

It smells like a critical behavior but, as Δ is not a physical quantity, we would not elaborate this point further right now.

At $m = 4.6705$ or at smaller values of mass, the solution exists, however. The profiles of the functions $\rho(z)$ and $R(z)$ for three values of mass: $m = 0.2$, $m = 2.0$, and $m = 4.6705$ in the whole interval $-\infty < z < \infty$ are plotted in Figs. 2,3. For small values of m, the solution approaches, as it should, the analytic solution (1.6) (with $\phi = \chi^{-2/3}$) found in Ref.[4], so that $\rho(z)$ and $R(z)$ stay constant.

In Fig. 4 we plotted the dependence of $R(0)$ on m. We see that $R(0)$ has a singular behavior at the phase transition point, but stays non-zero.

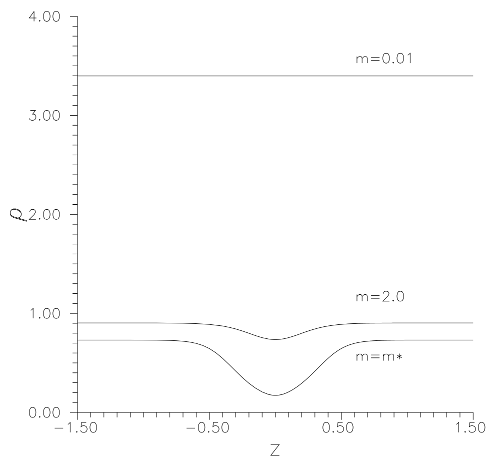

Figure 2: $\rho(z)$ for different masses.

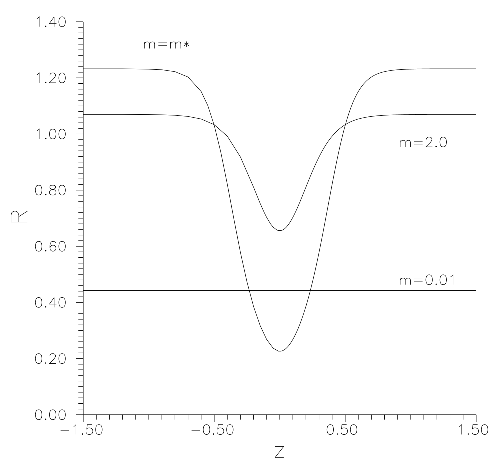

Figure 3: $R(z)$ for different masses.

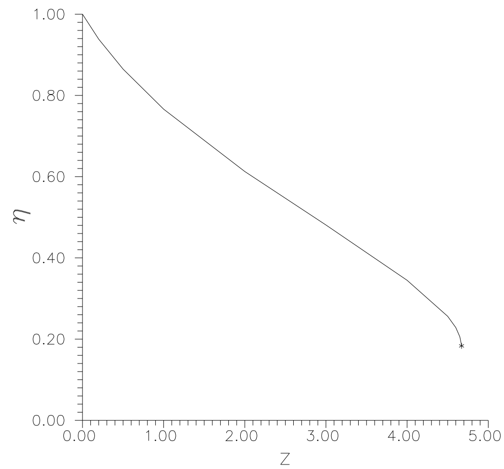

Figure 4: The ratio $\eta = R(0)/R(\infty)$ as a function of mass

3 Discussion.

We have found that the properties of the system are drastically changed at $m = 4.6705\ldots\Lambda$. It makes sense to express the result in invariant terms and to trade Λ for an invariant physical quantity such as the gluino condensate $\Sigma = |<\mathrm{Tr}\,\lambda^2>|$ in a chirally asymmetric vacuum. From Eqs.(1.10), (2.3), we obtain

$$\Sigma \;=\; \frac{16\pi^2}{\sqrt{3}} m^{1/2}\Lambda^{5/2} \tag{3.1}$$

Thus, the phase transition occurs at

$$m_* \;\approx\; 0.8\Sigma^{1/3} \tag{3.2}$$

The particular numerical value (3.2) was obtained by studying the effective TVY lagrangian with the potential (1.8) and the standard kinetic term $|\partial\phi|^2 + |\partial\chi|^2$. We cannot claim that the phase transition in the theory of interest (1.1) would occur at exactly the same value of mass. We believe, however, that the value of m_* in the supersymmetric QCD should be close to (3.2).

We hasten to comment that it is not a phase transition of habitual thermodynamic variety. In particular, the vacuum energy is zero both below and above the phase transition point — supersymmetry is never broken here. Hence $E_{vac}(m) \equiv 0$ is not singular at $m = m_*$.

Some similarities may be observed with the 2–dimensional Sine–Gordon model where the number of the states in the spectrum depends on the coupling constant β so that the states appear or disappear at some critical values of β [12]. May be a

more close analogy can be drawn with the $N = 2$ supersymmetric Yang–Mills theory. The spectrum of the system depends on the Higgs expectation value $u \; = < \mathrm{Tr}\, \varphi^2 >$. A study of the exact solution of the model due to Seiberg and Witten [13] displays the existence of a "marginal stability curve" in the complex u–plane [14]. When crossing this curve, the spectrum pattern is qualitatively changed.

To understand better the physics of the phase transition in supersymmetric QCD, one has to study in more details what happens in the region $m > m_*$. There are two logical possibilities:

- The complex domain wall solution may still exist at larger masses, but this solution cannot be BPS–saturated anymore. [4] In this case, it would be very interesting to study what happens in the limit $m \to \infty$ when the matter fields decouple and the theory is reduced to the pure $N = 1$ supersymmetric Yang–Mills theory. Do the domain walls interpolating between different chirally asymmetric vacua survive in this limit ? [5]
- The complex domain wall solution can disappear altogether at $m > m_*$.

The question of the existence and, if the solutions are there, of the properties of the complex domain walls at large masses is now under study. But in any case, it is clear now that there *is* no smooth transition between the weak coupling Higgs regime which is realized at small masses in chirally asymmetric phases and the strong coupling regime at large mass values.

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[4] For detailed discussion of a variety of model examples where the walls are or are not BPS–saturated, see [8]. Note that our observation contradicts a *conjecture* of Ref.[8] that the BPS solutions cannot appear or disappear at finite values of parameters in superpotential.

[5] The complex domain walls in the pure $N = 1$ SYM theory were discussed (assuming that they are there, of course) in recent [15] in the context of D–brane dynamics.

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